The Laplace Transform A Tutorial

The Laplace Transform is defined such that

Time

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\,dt$$

(Note f(t))

Note that this integral does not always converge

The inverse Laplace transform is written

$$\mathcal{L}^{-1}[F_{(s)}] = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\,e^{+st}\,ds$$

where

$$j = \sqrt{-1}$$

Actually, the implementation of these equations will be left as an exercise for the student.

However, in practice

$$s \equiv \frac{d}{dt} \quad \text{and} \quad \frac{1}{s} \equiv \int_{0-}^{t} dt$$

This allows the conversion of differential/integral equations to algebraic.

Problem 1.

Solve

$$y'(t) - 5y(t) = 0 \qquad\qquad y(\pi) = 2$$

Taking the Laplace Transform of both sides

$$\mathcal{L}\{y'(t) - 5y(t)\} = \mathcal{L}\{0\}$$

Note that 1) $\mathcal{L}$ is a linear operation
2) $\mathcal{L}\{0\}=0$

Then

$$\mathcal{L}\{y'(t)\} - 5\,\mathcal{L}\{y(t)\} = 0$$

By definition

$$\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y(t)\} - y(0)$$

Then

$$[s\,\mathcal{L}\{y(t)\}-y(0)]-5\,\mathcal{L}\{y(t)\}=0$$

Solving for $\mathcal{L}\{y(t)\}$

$$s\,\mathcal{L}\{y(t)\}-5\,\mathcal{L}\{y(t)\}=y(0)$$

$$[\mathcal{L}\{y(t)\}][s-5]=y(0)$$

$$\mathcal{L}\{y(t)\}=\frac{y(0)}{s-5}$$

Inverting this equation

$$y(t)=\mathcal{L}^{-1}\left\{\frac{y(0)}{s-5}\right\}$$

Note $\mathcal{L}^{-1}$ is also a linear operator so

$$y(t)=y(0)\,\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

From the table of Laplace Transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\}=e^{-at}$$

so $$a=-5$$

$$y(t)=y(0)\,\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}=y(0)e^{+5t}$$

so

$$y(t)=y(0)e^{5t}$$

From our initial conditions

$$y(\pi)=2$$

so

$$2=y(0)e^{5\pi}$$

or

$$y(0)=2e^{-5\pi}$$

Then

$$y(t) = 2e^{-5\pi}\, e^{5t}$$

$$y(t) = 2e^{5(t-\pi\}}$$

Problem 2.

Solve

$$y'(t)+by(t)=1 \qquad\qquad \text{where: } y(0)=0$$

$$\mathcal{L}\{y'(t)+by(t)\}=\mathcal{L}\{1\}$$

$$\mathcal{L}\{y'(t)\}+\mathcal{L}\{by(t)\}=\frac{1}{s}$$

$$s\mathcal{L}\{y(t)\}-y(0)+b\mathcal{L}\{y(t)\}=\frac{1}{s}$$

$$\mathcal{L}\{y(t)\}(s+b)=\frac{1}{s}+y(0)$$

$$\mathcal{L}\{y(t)\}=\frac{1}{(s+b)s}+0\cancel{\frac{1}{s+b}}$$

$$\mathcal{L}\{y(t)\} = \frac{1}{s(s+b)}$$

Using Partial Fractions

$$\frac{1}{s(s+b)}=\frac{A}{s}+\frac{B}{s+b}$$

Multiply both sides by s(s+b)

$$\frac{s(s+b)}{s(s+b)}=\frac{A(\cancel{s}(s+b))}{\cancel{s}}+\frac{B(s)\cancel{(s+b)}}{\cancel{s+b}}$$

$$1=As + Ab + Bs$$

Note this implies that

No "s"s on LHS

$$A\,b = 1 \quad \& \quad A + B = 0$$

$$A = \frac{1}{B}$$

$$B = -A$$

so

$$A = \frac{1}{b},\ B = \frac{-1}{b}$$

Thus (Substituting back in for A & B)

$$\mathcal{L}\{y(t)\} = \frac{1}{b}\frac{1}{s} + \frac{-1}{b}\left(\frac{1}{s+b}\right)$$

Taking the inverse

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{b}\left(\frac{1}{s}\right) - \frac{1}{b}\left(\frac{1}{s+b}\right)\right\}$$

$$= \frac{1}{b}\ \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{b}\ \mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}$$

From our table of Laplace Transforms

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s+b}\right) = e^{-bt}$$

Thus $$y(t) = \frac{1}{b}(1) - \frac{1}{b}e^{-bt}$$

$$= \frac{1}{b}\left(1 - e^{-bt}\right)$$

Problem 3.

Note Regarding $y''(t)$

Note that

$$y(t)'' = \frac{d(f(t))}{dt}$$

where:

$$f(t) = y'(t)$$

so
$$\mathcal{L}(y''(t)) = s\mathcal{L}\{f(t)\} - f(0)$$
$$= s\mathcal{L}\{y'(t)\} - y'(0)$$
$$= s(s\mathcal{L}\{y(t)\} - y(0)) - y'(0)$$
$$= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$$

Problem 4.

Solve the IVP

$$y''(t)+4y(t)=0 \qquad y(0)=2$$
$$y'(0)=2$$

$$\mathcal{L}\{y''(t)+4y(t)\}=\mathcal{L}\{0\}$$

$$s^2\ \mathcal{L}\{y(t)\}-sy(0)-y'(0)+4\mathcal{L}\{y(t)\}=0$$

$$(\mathcal{L}\{y(t)\})(s^2+4)=sy(0)+y'(0)$$

$$L\{\,y(t)\,\}=\frac{2s+2}{s^2+4}$$

Using $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s+2}{s^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

From our table

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+w^2}\right\}$$

$$w = 2 \qquad\qquad = \cos wt$$

$$= \cos 2t$$

And

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{w}{s^2+w^2}\right\}$$

$$w = 2 \qquad\qquad = \sin wt$$

$$= \sin 2t$$

Substituting back

$$y(t) = 2\cos 2t + \sin 2t$$

Problem 5

Finally consider the situation

$$\frac{d^2x}{dt^2} + \frac{2dx}{dt} + 2x = 2 \qquad\qquad \text{where } x(0) = x'(0) = 0$$

This can be written

$$x''(t) + 2x'(t) + 2x(t)$$

Starting with the LHS

$$\mathcal{L}\{x''(t)\} = s^2\,\mathcal{L}\{x(t)\} - s\,\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0}$$

And

$$\mathcal{L}\{2x'(t)\} = 2\,\mathcal{L}\,\{x'(t)\}$$

$$= 2\,s\mathcal{L}\,\{x(t)\} - 2\,\underbrace{x(0)}_{0}$$

And

$$\mathcal{L}\,\{2x(t)\} = 2\,\mathcal{L}\,\{x(t)\}$$

And the RHS Yields

$$\mathcal{L}\{2\} = 2\ \mathcal{L}\{1\}$$

$$= \frac{2}{s}$$

Substituting all of these back in yields

$$s^2\mathcal{L}\{x(t)\}+2s\ \mathcal{L}\{x(t)\}+2\mathcal{L}\{x(t)\}= \frac{2}{s}$$

$$(\mathcal{L}\{x(t)\})(s^2+2s+ 2) = \frac{2}{s}$$

$$\mathcal{L}\{x(t)\} = \frac{2}{s\left(s^2+2s+2\right)}$$

AN ASIDE

IN MATLAB for s^2+2s+2

Enter P=[1 2 2]

roots (P) $\Rightarrow$ -1+1j

s= -1+1j
or s+1-j=0

-1-1j

s= -1-1j
or s+1+j=0

so

$$\mathcal{L}\{x(t)\}=\frac{2}{s(s+1-j)(s+1+j)}$$

Using partial fraction expansion

$$\frac{2}{s\left(s^2+2s+2\right)}=\frac{A}{s}+\frac{B}{(s+1+j)}+\frac{C}{(s+1-j)}$$

Multiply both sides by s (s+1+j) (s+1-j) yielding

$$\frac{2(s)(s+1+j)(s+1-j)}{s\left(s^2+2s+2\right)}=A(s+1+j)(s+1-j)+B(s)(s+1-j)+C(s)(s+1+j)$$

$$2 = A\left(s^2 + 2s + 2\right) + B(s)\,(s + 1 - j) + C(s)\,(s + 1 + j)$$

$$2 = A(s^2 + 2s + 2) + B\left(s^2 + s(1 - j)\right) + C\left(s^2 + s(1 + j)\right)$$

or

$$2 = A\ s^2 + 2\ A\ s + 2A +$$

$$B\ s^2 + B\ s\ (1 - j) + 0 +$$

$$C\ s^2 + C\ s\ (1 + j)$$

This yields

$$2A = 2 \qquad \text{I}$$

$$2A + B(1-j) + C(1+j) = 0 \qquad \text{II}$$

$$A + B + C = 0 \qquad \text{III}$$

Thus 2A=2 (from I)

$$\boxed{A = 1}$$

Substituting into III

$$A + B + C = 0 \qquad \text{(from III)}$$

$$1 + B + C = 0$$

$$B = -1 - C$$

Substituting into II

$$2(1)+(-1-C)(1-j)+C(1+j)=0$$

$$\underbrace{2+(-1+j}-C+Cj)+C+Cj=0$$

$$(2-1+j)+(-C+C+Cj+Cj)=0$$

$$2\ C\ j = (-1 - j)$$

$$C = \frac{-1 - j}{2j}\left(\frac{j}{j}\right)$$

$$= \frac{-j - j^2}{2j^2}$$

$$= \frac{1 - j}{-2}$$

$$\boxed{C = \frac{-1 + j}{2}}$$

Since

$$A + B + C = 0$$

$$1 + B + \frac{-1 + j}{2} = 0$$

$$B = -1 + \frac{1 - j}{2}$$

$$B = \frac{-2}{2} + \frac{1 - j}{2}$$

$$\boxed{B = \frac{-1 - j}{2}}$$

Therefore:

$$\mathcal{L}\{x(t)\} = \frac{1}{s} + \left(\frac{-1 - j}{2}\right)\left(\frac{1}{s + 1 + j}\right) + \left(\frac{-1 + j}{2}\right)\left(\frac{1}{s + 1 - j}\right)$$

$$x(t) = L^{-1}\left\{\frac{1}{s}\right\} + \left(\frac{-1 - j}{2}\right)L^{-1}\left\{\frac{1}{s + 1 + j}\right\} + \left(\frac{-1 + j}{2}\right)L^{-1}\left\{\frac{1}{s + 1 - j}\right\}$$

so

$$x(t) = 1 + \left(\frac{-1 - j}{2}\right)\left(e^{(-1 - j)t}\right) + \left(\frac{-1 + j}{2}\right)\left(e^{(-1 + j)t}\right)$$

$$\left.\begin{aligned} e^{(-1-j)t} &= e^{-t} e^{-jt} \\ &= e^{-t}(\cos -t + j \sin -t) \\ &\text{and} \\ e^{(-1+j)t} &= e^{-t}(\cos t + j \sin t) \end{aligned}\right\} \qquad \underline{e^{(a+bj)t} = e^{at}(\cos bt + j \sin bt)}$$

Use this identity

Thus

$$x(t) = 1 + \left(\frac{-1-j}{2}\right)\left(e^{-t}\right)\left(\cos(-t) + j\sin(-t)\right) + \left(\frac{-1+j}{2}\right)\left(e^{-t}\right)\left(\cos t + j\sin t\right)$$

$$x(t) = 1 + \left(\frac{-1-j}{2}\right)\left(e^{-t}\right)\left(\cos(-t) + j\sin(-t)\right) + \left(\frac{-1+j}{2}\right)\left(e^{-t}\right)\left(\cos t + j\sin t\right)$$

$$\frac{2\left(x(t)-1\right)}{e^{-t}} = (-1-j)(\cos - t + j\sin - t) + (-1+j)(\cos t + j\sin t)$$

$$= -\cos - t - j\sin - t - j\cos - t - j^2 \sin - t$$

$$-\cos t - j\sin t + j\cos t + j^2 \sin t$$

NOTE sin -x= -sin x
cos -x= cos x

$$= -\cos t + j\sin t - j\cos t - \sin t$$

$$-\cos t - j\sin t + j\cos t - \sin t$$

$$= -2(\cos t + \sin t)$$

$$\frac{x(t)-1}{e^{-t}} = -1(\cos t + \sin t)$$

$$x(t)-1 = -e^{-t}(\cos t + \sin t)$$

$$x(t) = 1-e^{-t}(\cos t + \sin t)$$