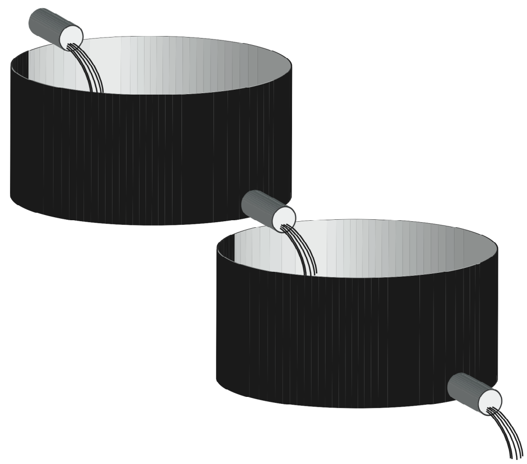
Transfer Function Development – Higher Order Systems
Surge Tanks in Series [1]

In the previous section we considered first order systems and the development of their transfer functions. In this section we will be considering higher order systems. The first case will be the response of two first order processes in series, two surge tanks in series.



In examining the above figure, it is important to note that the water can only flow from the upper tank to the lower tank. It can not flow in the opposite direction. Thus, the tanks are said to be *noniteracting.*

Recalling that the transfer function for a single surge tank is

$$G(s) = \frac{H(s)}{Q(s)}$$

where

$$\frac{H(s)}{Q(s)} = \frac{R}{\tau \bullet s + 1}$$

where $\tau = A \bullet R$
Q_1 = Flow into Tank 1

Since the outflow from tank 1 (the upper tank) is the inflow to tank 2 (the lower tank), the logical approach would be daisy chain the transfer functions. Unfortunately, the output variable for tank 1 is height, not flow. It critical that, if two transfer functions are to be multiplied, the output units from the first transfer function match the input units for the second transfer function. This problem can be solved by the creation of a third transfer function that performs the unit conversion. Thus, we have

$$G_{overall}(s) = \frac{H_1(s)}{Q_1(s)} \bullet \frac{Q_2(s)}{H_1(s)} \bullet \frac{H_2(s)}{Q_2(s)} = \frac{H_2(s)}{Q_1(s)}$$

where: Q_1 = Flow into tank 1
Q_2 = Flow into tank 2
H_1 = Head in tank 1
H_2 = Head in tank 2

Noting that from our definition of head and resistance,

$$Q_2 = \frac{H_1}{R_1}$$

we can write

$$\frac{Q_2}{H_1} = \frac{1}{R_1}$$

Note that it is possible to have a transfer function without "s." Such a transfer function is a pure gain and the response occurs without any lag. Then,

$$\frac{H_2(s)}{Q_1(s)} = \frac{R_1}{\tau_1 \bullet s + 1} \bullet \frac{1}{R_1} \bullet \frac{R_2}{\tau_2 \bullet s + 1}$$

or

$$\frac{H_2(s)}{Q_1(s)} = \frac{1}{\tau_1 \bullet s + 1} \bullet \frac{R_2}{\tau_2 \bullet s + 1}$$

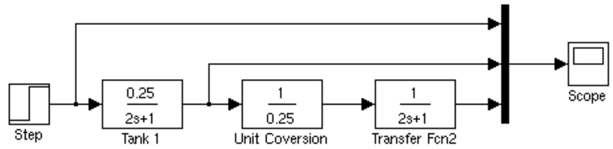
To determine the system's response to a unit step change in Q_1, we would multiply the overall transfer function by 1/s, yielding

$$H_2(s) = \frac{1}{s} \bullet \frac{1}{\tau_1 \bullet s + 1} \bullet \frac{R_2}{\tau_2 \bullet s + 1}$$

Using partial fractions to solve the equation results in a solution of

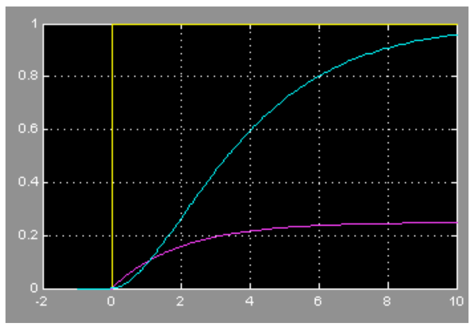
$$H_2(t) = R_2 \left[1 - \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \bullet \left(\frac{1}{\tau_2} \bullet e^{-t/\tau_1} - \frac{1}{\tau_1} \bullet e^{-t/\tau_2} \right) \right]$$

If this system is implemented in Simulink



where: $\tau_1 = 2$
$\tau_2 = 2$
$R_1 = 0.25$
$R_2 = 1$

This yields the following output



Note the difference in the shape of the curves for tank1 (pink line) and tank 2 (blue line) for the step input (yellow line). In addition, for tank 2, dH_2/dt is 0.

References

[1] D. R. Coughanowr and L. B. Koppel, *Process Systems Analysis and Control*. New York: Mc-Graw Hill Book Company, 1965.